# A STATISTICAL CANTING ANGLE MODEL USING HORIZONTAL WIND ENERGY SPECTRA

John Howard

RCA Astro-Electronics
Princeton, New Jersey

## Abstract

A statistical model estimating the mean and standard deviation of each dropsize canting angle, assuming Gaussian canting angle distributions, is presented. The model utilizes the one-dimensional energy spectrum of horizontal turbulence given by Smith[1] and by using the differential equation for the horizontal drop movement, it calculates the mean and standard deviation of each drop-size canting angle. Comparison with Saunders'[2] work shows good agreement.

## Introduction

Current interest in microwave propagation studies through precipitation particles has been prompted by proposals for terrestrial and satellite communication systems operating above 10 GHz. At these frequencies the presence of rain in the transmission medium causes attenuation and depolarization of the transmitted radiation. Both effects may represent a severe limitation on system performance and in particular, the depolarization effect is of considerable importance in the possibility of using two orthogonal polarizations as separate communication channels in future satellite and terrestrial communication systems.

The purpose of this paper is to report a statistical model relating turbulence to the reaction of the raindrop canting angle to this turbulence. It is assumed that in general a wind spectrum exists, and that within the turbulent boundary layer, the direction of the mean wind is constant[3] for the case of neutral or near neutral atmospheric conditions (i.e., purely mechanical turbulence), a case most appropriate in a precipitation environment[4,5]. Using the differential equation describing the horizontal drop movement, a canting angle 'Transfer Function' is developed. By applying the horizontal wind velocity spectrum as 'input' and by using Taylor's series[6], the 'response' of the drop (i.e., the mean and variance of its canting angle) is estimated.

## Theory

The differential equation for the horizontal drop movement is given by[7]

$$\frac{dV(t)}{dt} + \frac{g}{Vv} V(t) = \frac{g}{Vv} U(t) \tag{1}$$

where,

$V(t)$ is the horizontal drop velocity,
$Vv$ is the vertical drop velocity (assumed to be constant and equal to the terminal velocity in stagnant air),
$U(t)$ is the wind velocity at the position of the drop, and
$g$ is the gravitational constant.

Equation (1) is a linear differential equation of the first order and its solution for the mean wind velocity was given in Reference 8 for the case of neutral or near neutral conditions by using a logarithmic wind profile.

In addition to the mean horizontal wind velocity, there is a wind variation that can be described by,

$$U(t) = \int_0^\infty \hat{U}(f) \cos(\omega t + \theta'(f))\, df$$
$$= \int_0^\infty \frac{\hat{U}(f)}{2} \left(e^{(j\omega t + \theta'(f))} + e^{-(j\omega t + \theta'(f))}\right) df \tag{2}$$

where,

$\hat{U}(f)$ is the peak horizontal wind variation for a frequency f.

Similarly, the horizontal raindrop velocity variation can be described by,

$$V(t) = \int_0^\infty \hat{V}(f) \cos(\omega t + \phi'(f))\, df$$
$$= \int_0^\infty \frac{\hat{V}(f)}{2} \left(e^{(j\omega t + \phi'(f))} + e^{-(j\omega t + \phi'(f))}\right) df \tag{3}$$

where,

$\hat{V}(f)$ is the peak horizontal raindrop variation for a frequency f, and
$\phi(f)$ is the phase delay related to the response of the raindrop to wind fluctuations at frequency f, i.e., $\phi(f) = \theta'(f) - \phi'(f)$

Using equations (2) and (3) in equation (1) we obtain,

$$\frac{\hat{V}(f)}{2}\left(\frac{g}{Vv} + j\omega\right) e^{j(\omega t + \phi(f))} = \frac{\hat{U}(f)}{2} \frac{g}{Vv} e^{j\omega t} \tag{4a}$$

$$\frac{\hat{V}(f)}{2}\left(\frac{g}{Vv} - j\omega\right) e^{-j(\omega t + \phi(f))} = \frac{\hat{U}(f)}{2} \frac{g}{Vv} e^{-j\omega t} \tag{4b}$$

$$\hat{V}(f) \cos(\omega t + \phi(f))$$
$$= \frac{\hat{U}(f)}{2}\left(\frac{g/Vv}{g/Vv + j\omega} e^{j\omega t} + \frac{g/Vv}{g/Vv - j\omega} e^{-j\omega t}\right) \tag{5}$$

From Reference 7,

$$\tan\theta = \frac{U-V}{Vv} \tag{6}$$

where,

$\theta$ is the canting angle of a raindrop due to wind variations.

Then,

$$\tan\theta = \frac{\hat{U}(f)\cos\omega t - \hat{V}(f)\cos(\omega t + \phi(f))}{Vv}$$

$$= \frac{\hat{U}(f)}{2Vv}\left(\left(1 - \frac{g/Vv}{g/Vv+j\omega}\right)e^{j\omega t}+\left(1 - \frac{g/Vv}{g/Vv-j\omega}\right)e^{-j\omega t}\right)$$

$$= \frac{\hat{U}(f)}{Vv} \frac{1}{\sqrt{1+(g/\omega Vv)^2}}$$

$$\left(\frac{e^{j(\omega t+\tan^{-1}g/\omega Vv)} + e^{-j(\omega t+\tan^{-1}g/\omega Vv)}}{2}\right)$$

$$= \frac{\hat{U}(f)}{Vv} \frac{1}{\sqrt{1+(g/\omega Vv)^2}} \cos(\omega t+\tan^{-1} g/\omega Vv) \tag{7}$$

The variable $\frac{1}{Vv\sqrt{1+(g/\omega Vv)^2}}$ is the amplitude of a 'Transfer Function' relating $\tan\theta$ to a sinusoidal wind variation. The phase delay introduced by the transfer function is $\tan^{-1}(g/\omega Vv)$. The energy spectrum of $\tan\theta$, i.e., of the output, can be obtained by applying this transfer function to the horizontal wind velocity energy spectrum $S(f)$. Integrating this output spectrum over all frequencies, the variance of $\tan\theta$ can be calculated.

Thus,

$$\sigma^2_{\tan\theta} = \int_0^\infty |T.F.|^2 S(f)\,df$$

$$= \int_0^\infty \frac{1}{Vv^2(1+(g/\omega Vv)^2)} S(f)\,df \tag{8}$$

The one-dimensional energy spectrum $S(f)$ of turbulence can be expressed as[1],

$$S(f) = 0.15\ U^{2/3}\ \varepsilon^{2/3}\ f^{-5/3} \tag{9}$$

where,

$f$ is the frequency of wind variation,
$\varepsilon$ is the rate of turbulent energy dissipation, and
$U$ is the wind velocity at the height of interest.

In neutral or near neutral atmospheric conditions, the rate of dissipation is given by[1],

$$\varepsilon = U_*^2 \frac{dU}{dh} \tag{10}$$

where,

$U_*$ is the friction velocity, and
$h$ is the height of interest.

In the case where there is precipitation, the atmosphere is most likely to be neutral or near neutral. This is the case, since[5] (i) cloud cover will reduce incoming solar radiation so that turbulence will be losing energy, but (ii) the ground will be wet and hence, most of the available energy will go into evaporation, thus adding to the turbulent energy.

The combination of the above two conditions will produce energy equilibrium in the turbulence and thence, the atmospheric conditions will be neutral or near neutral. In addition, although the turbulence in the atmosphere is generally both convective and mechanical in origin, in high winds, even without any precipitation present, convective turbulence plays a relatively minor role[4]. The reason for this is that whereas mechanical turbulence rapidly increases with wind velocity, convective turbulence, if anything, tends to be damped out by the powerful mixing action caused by the mechanical turbulence; the latter prevents the necessary thermal instabilities from arising and tends to reduce the atmosphere to a state of neutral stability.

The mean and variance of the canting angle $\theta$ can be estimated by using the mean and variance of $\tan\theta$ and by employing the Taylor's series expansion assuming that near the mean value of $\tan\theta$, the probability density function of $\theta$, $f(\theta)$, is 'smooth'[6]. In the case where $f(\theta)$ is a Gaussian distribution, this is readily seen to be true. Thus, in general, if the mean value $\eta$ and variance $\sigma_x^2$ of the random variable $x$ are known, the random variable $g(x)$ has the following estimates of mean and variance

$$E\left[g(x)\right] \approx g(\eta) + g''(\eta)\frac{\sigma^2 x}{2} \tag{11}$$

$$\sigma_g^2(x) \approx g^2(\eta)+\left[(g'(\eta))^2+g(\eta)g''(\eta)\right]\sigma_x^2$$
$$-\left(g(\eta)+g''(\eta)\sigma_x^2\right)^2 \tag{12}$$

Using,

$$x = \tan\theta$$
$$\eta = \tan\theta_o$$
$$\sigma_x^2 = \sigma^2_{\tan\theta}$$
$$g(x) = \tan^{-1}(\tan\theta)$$

Using equations (11) and (12), we obtain the following estimates for the mean and variance values of the canting angle $\theta$,

$$E(\theta) \approx \tan^{-1}(\tan\theta_o)-\tan\theta_o\cos^4\theta_o\sigma^2_{\tan\theta} \tag{13}$$

$$\sigma_\theta^2 \approx \cos^4\theta_o\sigma^2_{\tan\theta}+\tan^2\theta_o\cos^8\theta_o\sigma^4_{\tan\theta} \tag{14}$$

where,

$$\tan\theta_o = \frac{U_*}{KVv}e^{(g/Vv^2)h}E_1(gh/Vv^2) \tag{15}$$

with,

$E_1(Z)$ the exponential integral

## Computations

In a paper by Saunders[2], the distribution of canting angles during two rainstorms is given using the images of 463 raindrops obtained with a raindrop camera[9,10,11] by personnel at the Illinois State Water Survey. In order to compare Saunders' results with the model presented in this paper, further information was needed in addition to the mean horizontal wind speed of 15 m/s and the precipitation rate of 28 mm/hr provided, namely, (i) the height of the raindrop camera, (ii) the type of the terrain where the measurements were taken, and (iii) the height of the anemometer.

The information supplied to the author[11,12] indicated that (i) the height of the camera was approximately 1.5 meters, (ii) the type of the terrain was 'a flat countryside with some form of vegetation,' and (iii) the height of the anemometer was assumed to be approximately 10 meters. Thus, the mean friction velocity $U_*$ was taken as equal to 1.303 m/s.

Assuming the above and using Equations (15) and (8) in Equations (13) and (14), the mean and standard deviation of the canting angle of each raindrop size were calculated. These are provided in Table 1.

Assuming a Gaussian canting angle distribution for each drop size and weighting each drop size distribution

TABLE 1. MEAN AND STANDARD DEVIATION OF CANTING ANGLE (Mean Horizontal Wind Velocity U = 15 m/s, Measured at Height h = 10 m, Height of Observations h = 1.5 m, Friction Velocity $U_*$ = 1.303 m/s)

| Drop Radius (m) | $E(\theta)$ (Degrees) | $\sigma_\theta$ (Degrees) |
|---|---|---|
| 0.00025 | 8.17 | 44.91 |
| 0.00050 | 21.57 | 25.76 |
| 0.00075 | 25.07 | 20.47 |
| 0.00100 | 26.45 | 17.89 |
| 0.00125 | 27.07 | 16.32 |
| 0.00150 | 27.32 | 15.42 |
| 0.00175 | 27.42 | 14.87 |
| 0.00200 | 27.46 | 14.54 |
| 0.00225 | 27.48 | 14.37 |
| 0.00250 | 27.48 | 14.28 |
| 0.00275 | 27.48 | 14.24 |
| 0.00300 | 27.48 | 14.23 |
| 0.00325 | 27.48 | 14.23 |

according to the number of drops for each size[13], the fraction of raindrop population with canting angles $\geqslant \theta$ and $\leqslant -\theta$ degrees, as in Saunders' Figure 2, were plotted. This is shown in Figure 1 compared with Saunders' graph for the 28 mm/hr precipitation rate and 15 m/s mean horizontal wind velocity.

## Discussion and Conclusions

In the present paper, a statistical canting angle model was presented that provides an estimate of the mean and standard deviation of the canting angle for the individual raindrop sizes. These estimates in turn can be used to provide estimates of mean and standard deviation of the cross-polarization discrimination values, as well as system outage times in both terrestrial and satellite links. In order to use the model, one needs to have knowledge of (i) the precipitation rate, (ii) the mean horizontal wind velocity at a reference height, and (iii) the type of terrain.

It is to be noted that near neutral atmospheric conditions are assumed, a situation most likely to exist when it is raining[5].

Compared with Saunders' measured canting angle distribution, the model shows good agreement. The differences between Saunders' and the results of Figure 1 in this paper are probably due to (i) inaccuracies introduced by the raindrop camera quoted by Saunders, (ii) the sample of 259 drops that Saunders used, to calculate its canting angle distribution, being too small for an accurate canting angle distribution to be drawn, and (iii) the equations presented in this paper, being approximate, not completely describing horizontal wind velocity variations and generally effects of atmospheric conditions on canting angles in a precipitation environment.

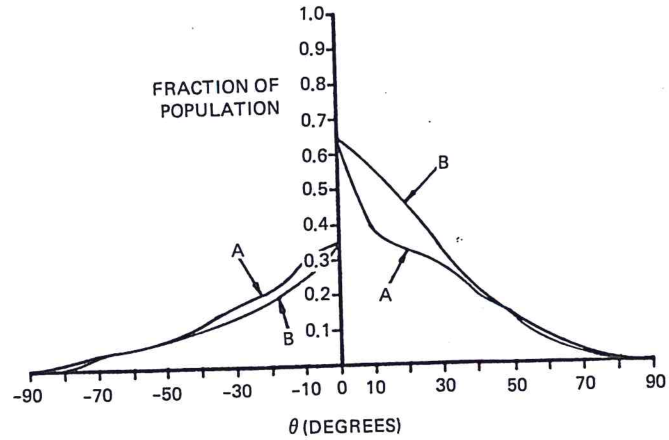


Figure 1. Fraction of Raindrop Population with Canting Angles $\geqslant\theta$ and $\leqslant-\theta$ (Curves B). Horizontal Wind Velocity U = 15 m/s, Height h = 1.5 m, Precipitation Rate 25 mm/hr. Comparison with Saunders' Results (Curves A) for 28 mm/hr Precipitation Rate.

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